$Y \overset{\text{def}}{=} \{y \in \mathscr{W} - \{0\} \mid \pi(y)$ has no a contractible neighbourhood$\}$ lies in a proper linear subspace of $\mathscr{W}$. But $T(G)$ acts irreducibly on $\mathscr{W}$ and the set Y is $T(G)$-invariant. Thus, $Y = \varnothing$. On the other hand, $\mathbf{P}(\mathscr{W})$ is proximal. Thus, Proposition 1.6(a) implies that the action of G on $\mathbf{P}(\mathscr{W})$ is strongly proximal.

(b) By Lemmas 3.2 and 3.4 there exist $r \in \mathbb{N}^+$, $D_1, \dots, D_r \subset \mathbf{GL}(\mathscr{W})$, and open subsets $U_1, \dots, U_r$ of $\mathbf{P}(\mathscr{W})$ such that $\mathbf{GL}(\mathscr{W}) = \bigcup_{1 \le j \le r} D_j \cdot GU_j = \mathbf{P}(\mathscr{W})$, and D_j is equicontinuous on U_j for each j, $1 \le j \le r$. By setting $C_j = T^{-1}(D_j)$, we see that condition (b) of Proposition 2.13 is satisfied. Thus (see Proposition 2.13), (a) above implies that the action of G on $\mathbf{P}(\mathscr{W})$ is mean proximal, completing the proof. □

(3.8) *Remark 1.* Since $\mathbf{H}^0$ coincides with the intersection of all algebraic subgroups of finite index in $\mathbf{H}$ and the subgroup $\{h \in \mathbf{H} \mid h\mathscr{W}' = \mathscr{W}'\}$ is algebraic for any linear subspace $\mathscr{W}' \subset \mathscr{W}$, condition (ii) in Theorem 3.7 is equivalent to the condition that

(ii′) the restriction of the representation T to any subgroup of finite index in G is irreducible.

Remark 2. Let M be a closed G-invariant subset of $\mathbf{P}(\mathscr{W})$. It is not hard to show that if the representation T is irreducible, then conditions (i) and (ii) in Theorem 3.7 are necessary for both the strong and mean proximality of the G-space M. However, this is false for an arbitrary representation T. The action of the group $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in k\right\}$ on $\mathbf{P}(k^2)$ provides an elementary example of this.

From Corollary 2.10 and Theorem 3.7(b) we deduce

(3.9) Corollary. *Let $\mu \in \mathscr{P}(G)$ be a measure with $\operatorname{supp} \mu = G$, M' a locally compact Hausdorff G-space, and $\nu' \in \mathscr{P}(M')$. Suppose that $\mu * \nu' = \nu'$ and conditions (i) and (ii) in Theorem 3.7 are satisfied.*

(a) *If $\varphi\colon M' \to \mathscr{P}(\mathbf{P}(\mathscr{W}))$ is a ν'-measurable G-equivariant map, then $\varphi(x) \in \delta_{\mathbf{P}(\mathscr{W})}$ for almost all (in measure ν') $x \in M'$;*

(b) *If $\varphi_1, \varphi_2\colon M' \to \mathbf{P}(\mathscr{W})$ are measurable G-equivariant maps, then $\varphi_1(x) = \varphi_2(x)$ for almost all $x \in M'$.*

In the next section we shall need the following

(3.10) Lemma. *Let $\mu \in \mathscr{P}(G)$ be a measure with $\operatorname{supp} \mu = G$ and $\nu \in \mathscr{P}(\mathbf{P}(\mathscr{W}))$. Suppose $\mu * \nu = \nu$ and condition (ii) of Theorem 3.7 is satisfied. Then for every proper linear subspace $\mathscr{W}'$ of $\mathscr{W}$, we have $\nu(\mathbf{P}(\mathscr{W}')) = 0$.*

Proof. We set

$$a(s) = \sup\{\nu(\mathbf{P}(\mathscr{W}')) \mid \mathscr{W}' \in \mathbf{Gr}_s(\mathscr{W})\},\ 1 \le s \le \dim \mathscr{W}, \text{ and}$$
$$l = \min\{s \in \mathbb{N}^+ \mid a(s) > 0\}.$$

It follows from the definition of the number l that $\nu(\mathbf{P}(\mathscr{W}_1) \cap \mathbf{P}(\mathscr{W}_2)) = 0$ for all distinct $\mathscr{W}_1, \mathscr{W}_2 \in \mathbf{Gr}_l(\mathscr{W})$. Thus, for every $\varepsilon > 0$ the set $\{\mathscr{W}' \in \mathbf{Gr}_l(\mathscr{W}) \mid \nu(\mathbf{P}(\mathscr{W}')) > \varepsilon\}$ is finite, and hence there exists $\mathscr{W}' \in \mathbf{Gr}_l(\mathscr{W})$ such that $\nu(\mathbf{P}(\mathscr{W}')) = a(l)$. Since $\mu * \nu = \nu$, it follows that

$$\nu(\mathbf{P}(\mathscr{W}')) = \int_G \nu(\mathbf{P}(g^{-1}\mathscr{W}'))d\mu(g).$$

But $\nu(\mathbf{P}(\mathscr{W}')) = a(l)$ implies that $\nu(\mathbf{P}(g^{-1}\mathscr{W}')) \le \nu(\mathbf{P}(\mathscr{W}'))$ for all $g \in G$. Thus, $\nu(\mathbf{P}(g^{-1}\mathscr{W}')) = \nu(\mathbf{P}(\mathscr{W}'))$ for almost all (in measure μ) $g \in G$. On the other hand, the support of the measure μ equals G and the set $\{\mathscr{W}'' \in \mathbf{Gr}_l(\mathscr{W}) \mid \mu(\mathbf{P}(\mathscr{W}'')) = \mu(\mathbf{P}(\mathscr{W}')) = a(l) > 0\}$ is finite. Therefore the set $\{g\mathscr{W}' \mid g \in G\} \subset \mathbf{Gr}_l(\mathscr{W})$ is finite. This is equivalent to the finiteness of the index of the subgroup $\{g \in G \mid g\mathscr{W}' = \mathscr{W}'\}$ in G. But condition (ii) of Theorem 3.7 is equivalent to (ii′) in 3.8. Thus, $\mathscr{W}' = \mathscr{W}$, and hence $l = \dim \mathscr{W}$. This completes the proof. □

4. Equivariant Measurable Maps to Algebraic Varieties

As in Section 3 of Chapter II, suppose that we are given a finite non-empty set A. For each $\alpha \in A$, we choose a local field k_α and a connected non-trivial semisimple k_α-group $\mathbf{G}_\alpha$. Denote by G the locally compact group $\prod_{\alpha \in A} \mathbf{G}_\alpha(k_\alpha)$. As in §3 of Chapter II, we remark that G is compactly generated. For $\alpha \in A$, choose a maximal k_α-split torus $\mathbf{S}_\alpha$ in $\mathbf{G}_\alpha$ and the minimal parabolic k_α-subgroup $\mathbf{P}_\alpha$ containing $\mathbf{S}_\alpha$. We set $S = \prod_{\alpha \in A} \mathbf{S}_\alpha(k_\alpha)$ and $P = \prod_{\alpha \in A} \mathbf{P}_\alpha(k_\alpha)$. We remark that G/P is compact, because $\mathbf{G}_\alpha(k_\alpha)/\mathbf{P}_\alpha(k_\alpha)$ is compact for all $\alpha \in A$ (see Corollary I.2.1.5).

Let Γ be a lattice in G and Λ a subgroup of $\operatorname{Comm}_G(\Gamma)$ containing Γ.

Thm 5.13

(4.1) Proposition. *Let $\mu_0 = \psi_0 \cdot \mu_G \in \mathscr{P}(G)$, where ψ_0 is a continuous function on G with compact support such that the set $\{g \in G \mid \psi_0(g) > 0$ and $\psi_0(g^{-1}) > 0\}$ generates the group G. Let $\nu_0 \in \mathscr{P}(G/P)$ with $\mu_0 * \nu_0 = \nu_0$. Then there is a measure $\mu \in \mathscr{P}(\Gamma)$ with $\operatorname{supp} \mu = \Gamma$ such that $\mu * \nu_0 = \nu_0$.*

First we shall prove the following two lemmas.

(A) Lemma. *If $g_1, g_2 \in G$, then there exist $n \in \mathbb{N}^+$ and $\varepsilon > 0$ such that $g_1\mu_0^{(n)} > \varepsilon g_2\mu_0$, where $\mu^{(n)} = \mu * \mu * \dots * \mu$ is the n-fold convolution of the measure μ.*

Proof. It follows from the conditions imposed on the measure μ_0 that for n sufficiently large, the density of measure $g_1\mu_0^{(n)}$ is positive on the support of the measure $g_2\mu_0$. This implies the assertion of the lemma, because the measure $g_2\mu_0$ has compact support and the densities of the measures $g_1\mu_0^{(n)}$ and $g_2\mu_0$ are continuous. □

Proof of Thm 5.13

(B) Lemma. *For $g \in G$ and $\gamma \in \Gamma$ there exist $\varepsilon = \varepsilon(g,\gamma) > 0$ and $\omega = \omega(g,\gamma) \in \mathbf{P}(G)$ such that $g\nu_0 = \varepsilon\gamma\nu_0 + (1-\varepsilon)\omega * \nu_0$.*

Proof. According to Lemma A, $g\mu_0^{(n)} > \varepsilon\gamma\mu_0$ for some $n \in \mathbb{N}^+$ and $\varepsilon > 0$. Since $\mu_0 * \nu_0 = \nu_0$ (and hence $\mu_0^{(n)} * \nu_0 = \nu_0$), it follows that

$$g\nu_0 = g\mu_0^{(n)} * \nu_0 = \varepsilon\gamma\mu_0 * \nu_0 + (g\mu_0^{(n)} - \varepsilon\gamma\mu_0) * \nu_0 =$$
$$= \varepsilon\gamma\nu_0 + (1-\varepsilon)\omega * \nu_0,$$

where $\omega = (1-\varepsilon)^{-1}(g\mu_0^{(n)} - \varepsilon\gamma\mu_0)$. This completes the proof. □

We now proceed to the proof of Proposition 4.1. For $g \in G$, we set $L(g) = \sup\{l \mid 0 \le l \le 1 \text{ and } g\nu_0 = l\mu' * \nu_0 + (1-l)\mu'' * \nu_0\}$, where $\mu' \in \mathbf{P}(\Gamma)$ with $\operatorname{supp}\mu' = \Gamma$ and $\mu'' \in P(G)$. It follows easily from Lemma B that $L(g) > 0$ for each $g \in G$ (the measure μ' should be selected in such a way that $\mu'(\{\gamma\}) < \varepsilon(g,\gamma)$ for all $\gamma \in \Gamma$). Clearly, $L(\gamma g) = L(g)$ for all $\gamma \in \Gamma$. From the equality

$$g\nu_0 = g\mu_0 * \nu_0 = \int_G g'\nu_0 d\mu_0(g')$$

we easily deduce that

$$L(g) \ge \int_G L(gg')d\mu_0(g').$$

By setting $L'(\Gamma g) = 1 - L(g)$, we obtain a function L' on $\Gamma \setminus G$ such that

$$L'(x) \le \int_G L'(xg)d\mu_0(g), \quad x \in \Gamma \setminus G. \tag{1}$$

Thus, by the Cauchy-Schwarz-Buniakowski inequality,

$$\int_{\Gamma\setminus G} L'(x)^2 d\mu_G(x) \le \int_{\Gamma\setminus G}\int_G L'(xg)^2 d\mu_0(g)d\mu_G(x) =$$
$$= \int_{\Gamma\setminus G} L'(x)^2\, d\mu_G(x),$$

the equality being attained if and only if $L'(xg) = L'(x)$ for almost all (relative to the measure $\mu_G * \mu_0$) pairs $(x,g) \in (\Gamma \setminus G) \times G$. But $\mu_0 = \psi_0\mu_G$ and $\operatorname{supp}\psi_0$ generates G. Thus the function $L'(x)$ is constant almost everywhere (relative to μ_G). It then follows from (1) and the positivity of the function $L(g)$ that there exists $\varepsilon > 0$ such that $L'(x) \le 1 - \varepsilon$ for all $x \in \Gamma \setminus G$ or, equivalently, $L(g) \ge \varepsilon$ for all $g \in G$. Choose a number $0 < l < \varepsilon$. Then, since $\mu * \nu_0 = \int_G g\nu_0 d\mu(g)$, for each measure $\sigma \in \mathscr{P}(G)$ there exist $\mu' \in \mathscr{P}(\Gamma)$ and $\mu'' \in \mathscr{P}(G)$ such that $\sigma * \nu_0 = l\mu' * \nu_0 + (1-l)\mu'' * \nu_0$ and $\operatorname{supp}\mu' = \Gamma$. Therefore

$$\nu_0 = l\mu_1 * \nu_0 + (1-l)[l\mu_2 * \nu_0 + (1-l)[l\mu_3 * \nu_0 + \dots,$$

where $\mu_i \in \mathscr{P}(\Gamma)$ with $\operatorname{supp}\mu_i = \Gamma$. Since $l + (1-l)l + (1-l)^2 l + \dots = 1$, the proof is complete. □